

THE STABILIZER OF IMMANANTS

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ABSTRACT. Immanants are homogeneous polynomials of degree n in n^2 variables associated to the irreducible representations of the symmetric group \mathfrak{S}_n of n elements. We describe immanants as trivial \mathfrak{S}_n modules and show that any homogeneous polynomial of degree n on the space of $n \times n$ matrices preserved up to scalar by left and right action by diagonal matrices and conjugation by permutation matrices is a linear combination of immanants. Building on works of M. Ant3nia Duffner [?] and Coelho, M. Purifica33o [?], we prove that for $n \geq 6$ the identity component of the stabilizer of any immanant (except determinant, permanent, and $\pi = (4, 1, 1, 1)$) is $\Delta(\mathfrak{S}_n) \ltimes T(GL_n \times GL_n) \ltimes \mathbb{Z}_2$, where $T(GL_n \times GL_n)$ is the group consisting of pairs of $n \times n$ diagonal matrices with the product of determinants 1, acting by left and right matrix multiplication, $\Delta(\mathfrak{S}_n)$ is the diagonal of $\mathfrak{S}_n \times \mathfrak{S}_n$, acting by conjugation, (\mathfrak{S}_n is the group of symmetric group.) and \mathbb{Z}_2 acts by sending a matrix to its transpose. Based on the work of Coelho, M. Purifica33o and Duffner, M. Ant3nia [?], we also prove that for $n \geq 5$ the stabilizer of the immanant of any non-symmetric partition (except determinant and permanent) is $\Delta(\mathfrak{S}_n) \ltimes T(GL_n \times GL_n) \ltimes \mathbb{Z}_2$.

1. Introduction

D.E. Littlewood [?] defined polynomials of degree n in n^2 variables generalizing the notion of determinant and permanent, called immanants, and are defined as follows:

Definition 1.1. For any partition $\pi \vdash n$, define a polynomial of degree n in matrix variables $(x_{ij})_{n \times n}$ associated to π as follows:

$$P_\pi := \sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\sigma) \prod_{i=1}^n x_{i\sigma(i)}$$

This polynomial is called the immanant associated to π .

Example 1.2. If $\pi = (1, 1, \dots, 1)$ then P_π is exactly the determinant of the matrix $(x_{ij})_{n \times n}$.

Example 1.3. If $\pi = (n)$ then P_π is the permanent $\sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n x_{i\sigma(i)}$.

Remark 1.4. Let E and F be \mathbb{C}^n . Since immanants are homogeneous polynomials of degree n in n^2 variables, we can identify them as elements in $S^n(E \otimes F)$ (Identify the space $E \otimes F$ with the space of $n \times n$ matrices). The space $S^n(E \otimes F)$ is a representation of $GL(E \otimes F)$, in particular, it is a representation of $GL(E) \times GL(F) \subset GL(E \otimes F)$. So we can use the representation theory of $GL(E) \times GL(F)$ to study immanants. The

explicit expression of an immanant P_π in $S^n(E \otimes F)$ is:

$$\sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\sigma) \prod_{i=1}^n e_i \otimes f_{\sigma(i)},$$

where \prod is interpreted as the symmetric tensor product.

In section 3 we remark that immanants can be defined as trivial \mathfrak{S}_n modules (Proposition 3.3). Duffner, M. Ant3nia found the system of equations determining the stabilizer of immanants (except determinant and permanent) for $n \geq 4$ in [?] in year 1994. 2 years later, Coelho, M. Purifica33o proved in [?] that if the system of equations in [?] has a solution, then permutations τ_1 and τ_2 in the system must be the same. Building on works of Duffner and Coelho, We prove the main results Theorem 1.5 and Theorem 1.7 of this paper in section 4.

Theorem 1.5. *Let π be a partition of $n \geq 6$ such that $\pi \neq (1, \dots, 1)$, $(4, 1, 1, 1)$ or (n) , then the identity component of the stabilizer of the immanant P_π is $\Delta(\mathfrak{S}_n) \rtimes T(GL_n \times GL_n) \rtimes \mathbb{Z}_2$, where $T(GL_n \times GL_n)$ is the group consisting of pairs of $n \times n$ diagonal matrices with the product of determinants 1, acting by left and right matrix multiplication, $\Delta(\mathfrak{S}_n)$ is the diagonal of $\mathfrak{S}_n \times \mathfrak{S}_n$, acting by conjugation, (\mathfrak{S}_n is the group of symmetric group.) and \mathbb{Z}_2 acts by sending a matrix to its transpose.*

Remark 1.6. It is well-known that Theorem 1.5 is true for permanent as well, but our proof does not recover this case.

Theorem 1.7. *Let $n \geq 5$ and let π be a partition of n which is not symmetric, that is, π is not equal to its transpose, and $\pi \neq (1, \dots, 1)$ or (n) , then the stabilizer of the immanant is $\Delta(\mathfrak{S}_n) \rtimes T(GL_n \times GL_n) \rtimes \mathbb{Z}_2$.*

Remark 1.8. One can compute directly from the system of equations determined by Duffner, M. Ant3nia for the case $n = 4$ and $\pi = (2, 2)$ and see that in this case, Theorem 1.7 fails, since there will be many additional components. For example,

$$C = \begin{pmatrix} e & -e & -e & e \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \frac{1}{e} & -\frac{1}{e} & \frac{1}{e} & -\frac{1}{e} \end{pmatrix}$$

stabilizes the immanant $P_{(2,2)}$, but it's not in the identity component.

2. Notations and preliminaries

- (1) E and F are n -dimensional complex vector spaces.
- (2) $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ are fixed basis of E and F , respectively.
- (3) \mathfrak{S}_n is the symmetric group on n elements. Given $\sigma \in \mathfrak{S}_n$, we can express σ as disjoint product of cycles, and we can denote the conjugacy class of σ by $(1^{i_1} 2^{i_2} \dots n^{i_n})$, meaning that σ is a disjoint product of i_1 1-cycles, i_2 2-cycles, ..., i_n n -cycles. Sometimes we might use (k_1, k_2, \dots, k_p) (where $n \geq k_1 \geq k_2 \geq \dots \geq k_p \geq 1$ and $\sum_{i=1}^p k_i = n$) to indicate the cycle type of σ . This notation means that σ contains a k_1 -cycle, a k_2 -cycle ... and a k_p -cycle.

- (4) $\pi \vdash n$ is a partition of n , we can write $\pi = (\pi_1, \dots, \pi_k)$ with $\pi_1 \geq \dots \geq \pi_k$.
- (5) π' is the conjugate partition of π .
- (6) M_π (sometimes we might use $[\pi]$ as well) is the irreducible representation of the symmetric group \mathfrak{S}_n corresponding to the partition π .
- (7) $S_\pi(E)$ is the irreducible representation of $GL(E)$ corresponding to π .
- (8) χ_π is the character of M_π .
- (9) Let V be a representation of $SL(E)$, then the weight-zero-subspace of V is denoted as V_0 .
- (10) For $S_\pi(E)$, we have a realization in the tensor product $E^{\otimes n}$ via the young symmetrizer $c_{\bar{\pi}} : S_\pi(E) \cong c_{\bar{\pi}}(E^{\otimes n})$, where $\bar{\pi}$ is any young tableau of shape π .
- (11) The action of \mathfrak{S}_n on E is the action of the Weyl group of $SL(E)$, that is, the permutation representation on \mathbb{C}^n .
- (12) Fix a young tableau $\bar{\pi}$, define $P_{\bar{\pi}} := \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ preserves each row of } \bar{\pi}\}$ and $Q_{\bar{\pi}} := \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ preserves each column of } \bar{\pi}\}$.
- (13) For a polynomial P in variables $(x_{ij})_{n \times n}$, denote the stabilizer of P in $GL(E \otimes F)$ by $G(P)$.

3. The description of immanants as modules

Consider the action of $T(E) \times T(F)$ on immanants, where $T(E), T(F)$ are maximal tori (diagonal matrices) of $SL(E), SL(F)$, respectively. For any $(A, B) \in T(E) \times T(F)$,

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{pmatrix}$$

For the immanant $P_\pi = \sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\sigma) \prod_{i=1}^n x_{i\sigma(i)}$, the action of (A, B) on P_π is given by

$$(A, B).P_\pi := \sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\sigma) \prod_{i=1}^n a_i b_{\sigma(i)} x_{i\sigma(i)} = P_\pi.$$

That is, immanants are in the $SL(E) \times SL(F)$ weight-zero-subspace of $S^n(E \otimes F)$. On the other hand, the decomposition of $S^n(E \otimes F)$ as $GL(E) \times GL(F)$ -modules:

$$S^n(E \otimes F) = \sum_{\lambda \vdash n} S_\lambda(E) \otimes S_\lambda(F).$$

Accordingly, we have a decomposition of the weight-zero-subspace:

$$(S^n(E \otimes F))_0 = \sum_{\lambda \vdash n} (S_\lambda(E))_0 \otimes (S_\lambda(F))_0.$$

Proposition 3.1. For $\lambda \vdash n$, $(S_\lambda(E))_0 \cong M_\lambda$ as \mathfrak{S}_n -modules.

Proof. See [?] page 272. □

Thus we can identify $(S_\lambda(E))_0 \otimes (S_\lambda(F))_0$ with $M_\lambda \otimes M_\lambda$ as an $\mathfrak{S}_n \times \mathfrak{S}_n$ module. Also, the diagonal $\Delta(\mathfrak{S}_n)$ of $\mathfrak{S}_n \times \mathfrak{S}_n$ is isomorphic to \mathfrak{S}_n , so $M_\lambda \otimes M_\lambda$ is a \mathfrak{S}_n -module. $M_\lambda \otimes M_\lambda$

is an irreducible $\mathfrak{S}_n \times \mathfrak{S}_n$ module, but it is reducible as and \mathfrak{S}_n -module, so that we can decompose it. Consider the action of \mathfrak{S}^n on $S^n(E \otimes F)$, let $\sigma \in \mathfrak{S}^n$, then:

$$\sigma.x_{ij} = x_{\sigma(i)\sigma(j)}.$$

So immanants are invariant under the action of \mathfrak{S}^n , hence are contained in the isotypic component of the trivial \mathfrak{S}_n representation of $\bigoplus_{\lambda \vdash n} (S_\lambda(E))_0 \otimes (S_\lambda(F))_0 = \bigoplus_{\lambda \vdash n} M_\lambda \otimes M_\lambda$.

Proposition 3.2. *As a \mathfrak{S}_n module, $M_\lambda \otimes M_\lambda$ contains only one copy of trivial representation.*

Proof. Denote the character of $\sigma \in \mathfrak{S}_n$ on $M_\lambda \otimes M_\lambda$ by $\chi(\sigma)$, and let $\chi_{trivial}$ be the character of the trivial representation. From the general theory of characters, it suffices to show that the inner product $(\chi, \chi_{trivial}) = 1$. First, the character $\chi(\sigma, \tau)$ of $(\sigma, \tau) \in \mathfrak{S}_n \times \mathfrak{S}_n$ on the module $M_\lambda \otimes M_\lambda$ is $\chi_\lambda(\sigma)\chi_\lambda(\tau)$. So in particular, the character χ of $M_\lambda \otimes M_\lambda$ on σ is $(\chi_\lambda(\sigma))^2$. Next,

$$\begin{aligned} (\chi, \chi_{trivial}) &= \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} \chi(\sigma) \overline{\chi_{trivial}(\sigma)} \right) \\ &= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (\chi_\lambda(\sigma))^2 \\ &= (\chi_\lambda, \chi_\lambda) \\ &= 1 \end{aligned}$$

since $\chi_{trivial}(\sigma) = 1, \forall \sigma \in \mathfrak{S}_n$. □

By the above proposition, $M_\lambda \otimes M_\lambda = \mathbb{C}_\lambda \oplus \dots$, where \mathbb{C}_λ means the unique copy of trivial representation in $M_\lambda \otimes M_\lambda$, and dots means other components in this module. Hence

$$P_\pi \in \bigoplus_{\lambda \vdash n} \mathbb{C}_\lambda.$$

We can further locate immanants:

Proposition 3.3. *Let P_π be the immanant associated to the partition $\pi \vdash n$. Assume \mathbb{C}_π is the unique copy of the trivial \mathfrak{S}_n -representation contained in $(S_\pi(E))_0 \otimes (S_\pi(F))_0$. Then: $P_\pi \in \mathbb{C}_\pi$.*

Before proving this proposition, we remark that it gives an equivalent definition of the immanant: P_π is the element of the trivial representation \mathbb{C}_π of $M_\pi \otimes M_\pi$ such that $P_\pi(Id) = \dim([\pi])$. For more information about this definition, see for example, [?].

Example 3.4. If $\pi = (1, \dots, 1)$, then $S_\pi E = \bigwedge^n E$, which is already a 1 dimensional vector space. If $\pi = (n)$, then $S_\pi E = S^n E$, in which there's only one (up to scale) weight zero vector $e_1 \circ \dots \circ e_n$.

Proof. Fix a partition $\pi \vdash n$, we want to show that the immanant P_π is in \mathbb{C}_π , but we know that P_π is in the weight-zero-subspace $(S^n(E \otimes F))_0 = \bigoplus_{\lambda \vdash n} (S_\lambda(E))_0 \otimes (S_\lambda(F))_0$. Since $(S_\lambda(E))_0 \otimes (S_\lambda(F))_0 \subset S_\lambda(E) \otimes S_\lambda(F)$, it suffices to show that $P_\pi \in S_\pi(E) \otimes S_\pi(F)$. Then it suffices to show that for any young symmetrizer $c_{\bar{\lambda}}$ not of the shape π , $c_{\bar{\lambda}} \otimes c_{\bar{\lambda}}(P_\pi) = 0$.

It suffices to check that $1 \otimes c_{\bar{\lambda}}(P_{\pi}) = 0$, since $c_{\bar{\lambda}} \otimes c_{\bar{\lambda}} = (c_{\bar{\lambda}} \otimes 1) \circ (1 \otimes c_{\bar{\lambda}})$. Express P_{π} as an element in $S^n(E \otimes F)$:

$$P_{\pi} = \sum_{\sigma \in \mathfrak{S}_n} \sum_{\tau \in \mathfrak{S}_n} \chi_{\pi}(\sigma) (\otimes_{i=1}^n e_{\tau(i)}) \otimes (\otimes_{i=1}^n f_{\sigma \circ \tau(i)})$$

The young symmetrizer $c_{\bar{\lambda}} = \sum_{p \in P_{\lambda}, q \in Q_{\lambda}} \text{sgn}(q) p q$. So

$$1 \otimes c_{\lambda}(P_{\pi}) = \sum_{\tau \in \mathfrak{S}_n} \otimes_{i=1}^n e_{\tau(i)} \otimes \left(\sum_{\substack{p \in P_{\lambda}, q \in Q_{\lambda} \\ \sigma \in \mathfrak{S}_n}} \chi_{\pi}(\sigma) \text{sgn}(q) \otimes_{i=1}^n f_{\sigma \cdot \tau \cdot q \cdot p(i)} \right)$$

In the above expression, $c_{\bar{\lambda}}$ acts on f_i 's. Now it suffices to show that:

$$\sum_{\substack{p \in P_{\lambda} \\ q \in Q_{\lambda} \\ \sigma \in \mathfrak{S}_n}} \chi_{\pi}(\sigma) \text{sgn}(q) \bigotimes_{i=1}^n f_{\sigma \cdot \tau \cdot q \cdot p(i)} = 0, \forall \tau \in \mathfrak{S}_n.$$

For any $\tau \in \mathfrak{S}_n$,

$$\begin{aligned} \sum_{\substack{p \in P_{\lambda} \\ q \in Q_{\lambda} \\ \alpha \in \mathfrak{S}_n}} \chi_{\pi}(\alpha \cdot \tau^{-1}) \text{sgn}(q) \bigotimes_{i=1}^n f_{\alpha \cdot q \cdot p(i)} &= \sum_{\gamma \in \mathfrak{S}_n} \sum_{\substack{p \in P_{\lambda} \\ q \in Q_{\lambda} \\ \alpha \in \mathfrak{S}_n \\ \alpha \cdot q \cdot p = \gamma}} \chi_{\pi}(\alpha \cdot \tau^{-1}) \text{sgn}(q) \bigotimes_{i=1}^n f_{\gamma(i)} \\ &= \sum_{\gamma \in \mathfrak{S}_n} \left[\sum_{\substack{p \in P_{\lambda} \\ q \in Q_{\lambda}}} \chi_{\pi}(\gamma \cdot p^{-1} \cdot q^{-1} \cdot \tau^{-1}) \text{sgn}(q) \right] \bigotimes_{i=1}^n f_{\gamma(i)}. \end{aligned}$$

Let $\sigma, \tau \in \mathfrak{S}_n$, $\sigma \cdot \tau = \sigma \cdot \tau \cdot \sigma^{-1} \cdot \sigma$, so $\sigma \cdot \tau = \tau' \cdot \sigma$, where τ' is conjugate to τ in \mathfrak{S}_n by σ . Therefore, we can rewrite the previous equation as:

$$\sum_{\gamma \in \mathfrak{S}_n} \left[\sum_{\substack{p \in P_{\lambda} \\ q \in Q_{\lambda}}} \chi_{\pi}(\tau^{-1} \cdot \gamma \cdot p^{-1} \cdot q^{-1}) \text{sgn}(q) \right] \bigotimes_{i=1}^n f_{\gamma(i)}.$$

Therefore, it suffices to show:

$$\sum_{\substack{p \in P_{\bar{\lambda}} \\ q \in Q_{\bar{\lambda}}}} \chi_{\pi}(\gamma \cdot p \cdot q) \text{sgn}(q) = 0, \forall \gamma \in \mathfrak{S}_n$$

This equality holds because the left hand side is the trace of $\gamma \cdot c_{\bar{\lambda}}$ as an operator on the space $\mathbb{C}\mathfrak{S}_n \cdot c_{\bar{\pi}}$, the group algebra of \mathfrak{S}_n , which is a realization of M_{π} in $\mathbb{C}\mathfrak{S}_n$, but this operator is in fact zero:

$$\forall \sigma \in \mathfrak{S}_n, \gamma \cdot c_{\bar{\lambda}} \cdot \sigma \cdot c_{\bar{\pi}} = \gamma \cdot \sigma \cdot c_{\bar{\lambda}'} \cdot c_{\bar{\pi}} = 0,$$

where $\bar{\lambda}'$ is the young tableau of shape λ' which is conjugate to $\bar{\lambda}$ by σ . This implies that $c_{\bar{\lambda}}$ and $c_{\bar{\pi}}$ are of different type, hence $c_{\bar{\lambda}} \cdot c_{\bar{\pi}} = 0$, in particular, the trace of this operator is 0. Therefore, $P_{\pi} \in S_{\pi}(E) \otimes S_{\pi}(F)$. \square

Corollary 3.5. *If a homogeneous polynomial Q of degree n is preserved by the group $\Delta(\mathfrak{S}_n) \times T(E) \times T(F)$, then Q is a linear combination of immanants. Furthermore, immanants are linearly independent and form a basis of the space of all homogeneous degree n polynomials preserved by $\Delta(\mathfrak{S}_n) \times T(E) \times T(F)$.*

Proof. If Q is preserved by $\mathfrak{S}_n \times T(GL(E) \times GL(F))$, then Q is in $\bigoplus_{\lambda \vdash n} \mathbb{C}_\lambda$. By the proposition, immanants form a basis of $\bigoplus_{\lambda \vdash n} \mathbb{C}_\lambda$, the corollary follows. \square

4. The stabilizer of immanant

Next, we study the stabilizer of immanants in the group $GL(E \otimes F)$.

Example 4.1. : For $\pi = (1, 1, \dots, 1)$ and $\pi = (n)$, $G(P_\pi)$ are well-known: If $\pi = (1, 1, \dots, 1)$, then $G(P_\pi) = S(GL(E) \times GL(F)) \ltimes \mathbb{Z}_2$, and if $\pi = (n)$, then $G(P_\pi) = \mathfrak{S}_n \times \mathfrak{S}_n \ltimes T(GL(E) \times GL(F)) \ltimes \mathbb{Z}_2$, where $S(GL(E) \times GL(F))$ is a subgroup of $GL(E) \times GL(F)$ consisting of pairs (A, B) with $\det(A)\det(B) = 1$, $T(GL(E) \times GL(F))$ is the pair of diagonal matrices with the product of determinants 1, and $\ltimes \mathbb{Z}_2$ means that we are allowed to take the transpose of matrices. For the stabilizer of determinant, see G.Frobenius [?]. For the stabilizer of permanent, see Botta [?].

Assume $C = (c_{ij})$ and $X = (x_{ij})$ are $n \times n$ matrices. Denote the torus action of C on X by $C * X = Y$, where $Y = (y_{ij})$ is an $n \times n$ matrix with entry $y_{ij} = c_{ij}x_{ij}$. Note that the torus action is just the action of the diagonal matrices in $End(E \otimes F)$ on the vector space $E \otimes F$. To find $G(P_\pi)$, we need the following result from [?]:

Theorem 4.2 ([?]). *Assume $n \geq 4$, $\pi \neq (1, 1, \dots, 1)$ and (n) . A linear transformation $T \in GL(E \otimes F)$ preserves the immanant P_π iff $T \in T(GL(E \otimes F)) \ltimes \mathfrak{S}_n \ltimes \mathfrak{S}_n \ltimes \mathbb{Z}_2$, and satisfies the relation:*

$$\chi_\pi(\sigma) \prod_{i=1}^n c_{i\sigma(i)} = \chi_\pi(\tau_2 \sigma \tau_1^{-1}),$$

where σ runs over all elements in \mathfrak{S}_n , $T(GL(E \otimes F))$ is the torus of $GL(E \otimes F)$, acting by the torus action described above, \mathfrak{S}_n is the symmetric group in n elements, acting by left and right multiplication, and \mathbb{Z}_2 sending a matrix to its transpose.

sketch. Step 1: Let π be a fixed partition of n . Define a subset \mathfrak{A} of the set $M_n(\mathbb{C})$ of n by n matrices as follows, $X^{(n-1)} := \{A \in M_n(\mathbb{C}) : \text{degree of } P_\pi(xA + B) \leq 1, \text{ for every } B \in M_n(\mathbb{C})\}$. Geometrically, $X^{(n-1)}$ is the most singular locus of $IM_\pi = 0$. If A is in $X^{(n-1)}$, and T preserves P_π (it can be shown that T is invertible), then we have that $T(A) \in X^{(n-1)}$, since the preserver of the hypersurface $IM_\pi = 0$ will preserve the most singular locus as well.

Step 2: Characterize the set $X^{(n-1)}$. To do this, first define a subset R_i (resp. R^i) of $M_n(\mathbb{C})$, consisting of matrices that have nonzero entries only in i -th row (resp. column). Then one proves that $A \in X^{(n-1)}$ if and only if it is in one of the forms:

- (1) R_i or R^i for some i .
- (2) The nonzero elements are in the 2×2 submatrix $A[i, h \mid i, h]$, and

$$\chi_\pi(\sigma) a_{ii} a_{hh} + \chi_\pi(\tau) a_{ih} a_{hi} = 0$$

for every σ and τ satisfying $\sigma(i) = i$, $\tau(h) = h$ and $\tau = \sigma(ih)$.

- (3) $\pi = (2, 1, \dots, 1)$ and there are complementary sets of indices $\{i_1, \dots, i_p\}$, $\{j_1, \dots, j_q\}$ such that the nonzero elements are in $A[i_1, \dots, i_p \mid j_1, \dots, j_q]$ and the rank of $A[i_1, \dots, i_p \mid j_1, \dots, j_q]$ is one.
- (4) $\pi = (n-1, 1)$, the nonzero elements are in a 2 by 2 submatrix $A[u, v \mid r, s]$, and the permanent of this submatrix is zero.

Step 3: Characterize T by sets R_i and R^j . □

We will start from this theorem. From this theorem, we know that $G(P_\pi)$ is contained in the group $\mathfrak{S}_n \times \mathfrak{S}_n \times \mathbb{C}^{n^2} \rtimes \mathbb{Z}^2$, and subject to the relation in Theorem 4.2.

Remark 4.3. In the equation in the Theorem 4.2, $c_{ij} \neq 0, \forall 1 \leq i, j \leq n$, since the stabilizer of P_π is a group.

Now instead of considering n^2 parameters, we can consider $n!$ parameters, *i.e.*, consider the stabilizer of immanant in the bigger monoid $\mathfrak{S}_n \times \mathfrak{S}_n \times \mathbb{C}^{n!} \rtimes \mathbb{Z}^2$. We can ignore the \mathbb{Z}^2 -part of this monoid. The action of the monoid on the weight zero space of $S^n(E \otimes F)$ spanned by monomials $x_{1\sigma(1)}x_{2\sigma(2)} \cdots x_{n\sigma(n)}$, $\sigma \in \mathfrak{S}_n$ is:

$$(\tau_1, \tau_2, (c_\sigma)_{\sigma \in \mathfrak{S}_n}) \cdot (x_{1\sigma(1)}x_{2\sigma(2)} \cdots x_{n\sigma(n)}) = c_\sigma x_{\tau_1(1)\tau_2\sigma(1)}x_{\tau_1(2)\tau_2\sigma(2)} \cdots x_{\tau_1(n)\tau_2\sigma(n)}.$$

Proposition 4.4. *The stabilizer of P_π in $\mathfrak{S}_n \times \mathfrak{S}_n \times \mathbb{C}^{n!}$ is determined by equations*

$$(1) \quad c_{\tau_2^{-1}\tau_1\sigma} \chi_\pi(\tau_2^{-1}\tau_1\sigma) = \chi_\pi(\sigma), \forall \sigma \in \mathfrak{S}_n$$

Proof. The action of this monoid on P_π is:

$$\begin{aligned} (\tau_1, \tau_2, (c_\sigma)_{\sigma \in \mathfrak{S}_n}) \cdot P_\pi &= \sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\sigma) c_\sigma \prod_{i=1}^n x_{\tau_1(i), \tau_2\sigma(i)} \\ &= \sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\sigma) c_\sigma \prod_{i=1}^n x_{i, \tau_2\sigma\tau_1^{-1}(i)} \\ &= \sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\tau_2^{-1}\sigma\tau_1) c_{\tau_2^{-1}\sigma\tau_1} \prod_{i=1}^n x_{i, \sigma(i)} \\ &= \sum_{\sigma \in \mathfrak{S}_n} \chi_\pi(\tau_2^{-1}\tau_1(\tau_1^{-1}\sigma\tau_1)) c_{\tau_2^{-1}\tau_1(\tau_1^{-1}\sigma\tau_1)} \prod_{i=1}^n x_{i, \sigma(i)} \end{aligned}$$

If $(\tau_1, \tau_2, (c_\sigma))$ stabilizes P_π , then

$$\chi_\pi(\tau_2^{-1}\tau_1(\tau_1^{-1}\sigma\tau_1)) c_{\tau_2^{-1}\tau_1(\tau_1^{-1}\sigma\tau_1)} = \chi_\pi(\sigma) = \chi_\pi(\tau_1^{-1}\sigma\tau_1), \forall \sigma \in \mathfrak{S}_n.$$

Therefore, we have: $c_{\tau_2^{-1}\tau_1\sigma} \chi_\pi(\tau_2^{-1}\tau_1\sigma) = \chi_\pi(\sigma), \forall \sigma \in \mathfrak{S}_n$. □

Our next task is to find τ_1 and τ_2 , such that the equation (1) has solution for $(c_\sigma)_{\sigma \in \mathfrak{S}_n}$. For convenience, in the equation (1), set $\tau_2^{-1}\tau_1 = \tau$, so we get a new equation:

$$(2) \quad c_{\tau\sigma} \chi_\pi(\tau\sigma) = \chi_\pi(\sigma), \forall \sigma \in \mathfrak{S}_n$$

Lemma 4.5. *If the equation (2) has a solution then $\tau \in \mathfrak{S}_n$ satisfies:*

- (1) *If $\chi_\pi(\sigma) = 0$, then $\chi_\pi(\tau\sigma) = 0$;*
- (2) *If $\chi_\pi(\sigma) \neq 0$, then $\chi_\pi(\tau\sigma) \neq 0$;*

Proof. Clear. □

Definition 4.6. : For a fixed partition $\pi \vdash n$, define:

$$P := \{\sigma \in \mathfrak{S}_n \mid \chi_\pi(\sigma) = 0\},$$

$$Q := \{\sigma \in \mathfrak{S}_n \mid \chi_\pi(\sigma) \neq 0\}.$$

$$G := (\cap_{\sigma \in P} P\sigma) \cap (\cap_{\sigma \in Q} Q\sigma).$$

Lemma 4.7. *The equation (2) has a solution, then $\tau \in G$.*

Proof. It suffices to show that the 2 conditions in Lemma 4.5 imply $\tau \in G$. If τ satisfies conditions 1 and 2, then

$$\tau\sigma \in P, \forall \sigma \in P; \tau\sigma' \in Q, \forall \sigma' \in Q,$$

therefore

$$\tau \in P\sigma^{-1}, \forall \sigma \in P; \tau \in Q\sigma'^{-1}, \forall \sigma' \in Q,$$

so

$$\tau \in G.$$

□

Example 4.8. We can compute G directly for small n . If $n = 3$, then we have three representations $M_{(3)}$, $M_{(1,1,1)}$, $M_{(2,1)}$, $G = \mathfrak{S}_3, A_3, A_3$, respectively. If $n = 4$, then we have five representations $M_{(4)}$, $M_{(1,1,1,1)}$, $M_{3,1}$, $M_{(2,1,1)}$, $M_{(2,2)}$, Here $G = \mathfrak{S}_4, A_4, \{(1), (12)(34), (13)(24), (14)(23)\}, \{(1), (12)(34), (13)(24), (14)(23)\}, A_4$, respectively. If the partition $\pi = (1, 1, \dots, 1)$, then $G = A_n$. If the partition $\pi = (n)$, then $G = \mathfrak{S}_n$. Note that in these examples, G is a normal subgroup. In fact, this holds in general.

The following proposition is due to Coelho, M. Purificação in [?], using Murnagham-Nakayama Rule. One can give a different proof using Frobenius character formula for cycles (see, for example, [?]).

Proposition 4.9 ([?]). *For any $n \geq 5$ and partition $\pi \vdash n$, G is a normal subgroup of \mathfrak{S}_n . Moreover, if $\pi \neq (1, 1, \dots, 1)$ or (n) , then G is the trivial subgroup (1) of \mathfrak{S}_n .*

Sketch. It is easy to show that G is a normal subgroup of \mathfrak{S}_n . And then one can prove $G \neq \mathfrak{S}_n$ by computing character χ_π . Then assume $G = A_n$, one can show that $Q = A_n$ and $P = \mathfrak{S}_n - A_n$. If such a partition π exists, then it must be symmetric. But then one can construct cycles σ contained in A_n case by case (using Murnagham-Nakayama rule or Frobenius's character formula) such that $\chi_\pi(\sigma) = 0$. It contradicts that $Q = A_n$. □

Let's return to the equation (see Theorem 4.2):

$$\chi_\pi(\sigma) \prod_{i=1}^n c_{i\sigma(i)} = \chi_\pi(\tau_2 \sigma \tau_1^{-1}) \quad \forall \sigma \in \mathfrak{S}_n$$

By Proposition 4.9, we can set $\tau_1 = \tau_2$ in the above equation then we have equations for c_σ 's:

$$(3) \quad \prod_{i=1}^n c_{i\sigma(i)} = 1 \quad \forall \sigma \in \mathfrak{S}_n, \text{ with } \chi_\pi(\sigma) \neq 0.$$

So elements in $G(P_\pi)$ can be expressed as triples $(\tau, \tau, (c_{ij}))$ where matrices (c_{ij}) is determined by equation(3).

Remark 4.10. The coefficients of those linear equations are $n \times n$ permutation matrices. If we ignore the restriction $\chi_\pi(\sigma) \neq 0$, then we get all $n \times n$ permutation matrices.

Lemma 4.11. *The permutation matrices span a linear space of dimension $(n-1)^2 + 1$ in $Mat_{n \times n} \cong \mathbb{C}^{n^2}$.*

Proof. Consider the action of \mathfrak{S}_n on \mathbb{C}^n by permuting the entries, so $\sigma \in \mathfrak{S}_n$ is an element in $End(\mathbb{C}^n)$, corresponding to a permutation matrix, and vice versa. Now as \mathfrak{S}_n modules, $\mathbb{C}^n \cong M_{(n-1,1)} \oplus \mathbb{C}$, where \mathbb{C} is the trivial representation of \mathfrak{S}_n . So we have a decomposition of vector spaces:

$$\begin{aligned} End(\mathbb{C}^n) &\cong End(M_{(n-1,1)} \oplus \mathbb{C}) \\ &\cong End(M_{(n-1,1)}) \oplus End(\mathbb{C}) \oplus Hom(M_{(n-1,1)}, \mathbb{C}) \oplus Hom(\mathbb{C}, M_{(n-1,1)}). \end{aligned}$$

Since \mathbb{C} and $M_{(n-1,1)}$ are \mathfrak{S}_n modules, $\mathfrak{S}_n \hookrightarrow End(M_{(n-1,1)}) \oplus End(\mathbb{C})$. Note that $dim(End(M_{(n-1,1)}) \oplus End(\mathbb{C})) = (n-1)^2 + 1$, so it suffices to show that \mathfrak{S}_n will span $End(M_{(n-1,1)}) \oplus End(\mathbb{C})$, but this is not hard to see because we have an algebra isomorphism:

$$\mathbb{C}[\mathfrak{S}_n] \cong \bigoplus_{\lambda \vdash n} End(M_\lambda)$$

Hence, we have:

$$\mathbb{C}[\mathfrak{S}_n] \twoheadrightarrow End(M_{(n-1,1)}) \oplus End(\mathbb{C}).$$

□

Remark 4.12. Lemma 4.11 shows that the dimension of the stabilizer of an immanant is at least $2n-2$. We will show next that for any partition π of $n \geq 5$ except $(1, \dots, 1)$ and (n) , the dimension of the stabilizer $G(P_\pi)$ is exactly $2n-2$.

We compute the Lie algebra of the stabilizer of $G(P_\pi)$. Since $G(P_\pi) \subset GL(E \otimes F)$, the Lie algebra of $G(P_\pi)$ is a subalgebra of $\mathfrak{gl}(E \otimes F)$. We have the decomposition of $\mathfrak{gl}(E \otimes F)$

$$\begin{aligned} \mathfrak{gl}(E \otimes F) &= End(E \otimes F) \\ &\cong (E \otimes F)^* \otimes (E \otimes F) \\ &\cong E^* \otimes E \otimes F^* \otimes F \\ &\cong (\mathfrak{sl}_R(E) \oplus Id_E \oplus T(E)) \otimes (\mathfrak{sl}_R(F) \oplus Id_F \oplus T(F)) \\ &\cong \mathfrak{sl}_R(E) \otimes \mathfrak{sl}_R(F) \oplus \mathfrak{sl}_R(E) \otimes Id_F \oplus Id_E \otimes \mathfrak{sl}_R(F) \oplus Id_E \otimes Id_F \\ &\quad \oplus T(E) \otimes T(F) \oplus T(E) \otimes Id_F \oplus Id_E \otimes T(F) \end{aligned}$$

Where $\mathfrak{sl}_R(E)$ is the root space of $\mathfrak{sl}(E)$, $T(E)$ is the torus of $\mathfrak{sl}(E)$, and Id_E is the space spanned by identity matrix. The similar notation is for F . We will show that the Lie algebra of $\{C \in M_{n \times n} \mid P_\pi(C * X) = P_\pi(X)\}$ is $T(E) \otimes Id_F \oplus Id_E \otimes T(F)$. Let $\{e_i \mid i = 1, \dots, n\}$ be a fixed basis of E and $\{\alpha^i \mid i = 1, \dots, n\}$ be the dual basis. Let $H_{1i}^E = \alpha^1 \otimes e_1 - \alpha^i \otimes e_i$. Then $\{H_{1i} \mid i = 2, \dots, n\}$ is a basis of $T(E)$. We use H^F for F and define $A_{ij} = H_{1i}^E \otimes H_{1j}^F$ for all $i \geq 2, j \geq 2$.

Now consider the action of A_{ij} on variable x_{pq} .

$$(4) \quad C_{p,q}^{i,j} := A_{ij}(x_{pq}) = (\delta_p^1 - \delta_p^i)(\delta_q^1 - \delta_q^j)$$

$$C_{p,q}^{i,j} = \begin{cases} 1, & p = q = 1 \\ -1, & p = i, q = 1 \\ -1, & p = 1, q = j \\ 1, & p = i, q = j \\ 0, & \text{otherwise} \end{cases}$$

Equation (3) implies that the matrices $C = (c_{ij})$ that stabilize P_π is contained in the torus of $GL(E \otimes F)$, hence the Lie algebra of the set of such matrices is contained in the torus of $\mathfrak{gl}(E \otimes F)$, that is, it is contained in $t := Id_E \otimes Id_F \oplus T(E) \otimes T(F) \oplus T(E) \otimes Id_F \oplus Id_E \otimes T(F)$. Now let L be an element of t , then L can be expressed as the linear combination of A_{ij} 's and $Id_E \otimes Id_F$. Hence:

$$L = (a Id_E \otimes Id_F + \sum_{i,j>1} a_{ij} A_{ij})$$

for some $a, a_{ij} \in \mathbb{C}$.

Then

$$(5) \quad L(x_{pq}) = \begin{cases} (a + \sum_{i,j>1} a_{ij})x_{11}, & p = q = 1 \\ (a - \sum_{i>1} a_{iq})x_{1q}, & p = 1, q \neq 1 \\ (a - \sum_{j>1} a_{pj})x_{p1}, & p \neq 1, q = 1 \\ (a + a_{pq})x_{pq}, & p \neq 1, q \neq 1 \end{cases}$$

Now for a permutation $\sigma \in \mathfrak{S}_n$, the action of L on the monomial $x_{1\sigma(1)} x_{2\sigma(2)} \dots x_{n\sigma(n)}$ is: if $\sigma(1) = 1$,

$$(6) \quad L(\prod_{p=1}^n x_{p\sigma(p)}) = (na + \sum_{i,j>1} a_{ij} + \sum_{p=2}^n a_{p\sigma(p)}) \prod_{p=1}^n x_{p\sigma(p)}.$$

if $\sigma(1) \neq 1$ and $\sigma(k) = 1$,

$$(7) \quad L(\prod_{p=1}^n x_{p\sigma(p)}) = (na + \sum_{p \neq 1, k} a_{p\sigma(p)} - \sum_{i>1} a_{i\sigma(1)} - \sum_{j>1} a_{kj}) \prod_{p=1}^n x_{p\sigma(p)}.$$

Lemma 4.13. *For any solution of the system of linear equations*

$$(8) \quad a_{ij} + a_{jk} + a_{km} = a_{ik} + a_{kj} + a_{jm}, \text{ where } \{i, j, k, m\} = \{2, 3, 4, 5\}$$

$$(9) \quad a_{ij} + a_{jk} = a_{ij'} + a_{j'k}, \text{ where } \{i, j, k, j'\} = \{2, 3, 4, 5\}$$

There exists a number λ such that for any permutation μ of the set $\{2, 3, 4, 5\}$ moving l elements,

$$(10) \quad \sum_{\substack{i=2 \\ \mu(i) \neq i}}^5 a_{i\mu(i)} = l\lambda$$

Proof. check by solving this linear system. \square

Lemma 4.14. let $n \geq 6$ be an integer, π be a fixed partition of n , which is not $(1, \dots, 1)$ or (n) . Assume that there exists a permutation (so a conjugacy class) $\tau \in \mathfrak{S}_n$ such that:

- (1) $\chi_\pi(\tau) \neq 0$;
- (2) τ contains a cycle moving at least 4 numbers;
- (3) τ fixes at least 1 number.

Also assume that $L(P_\pi) = 0$. Then under the above assumptions, $a = a_{ij} = 0$ for all $i, j > 1$.

Proof. $L(P_\pi) = 0$ means that $L(\prod_{p=1}^n x_{p\sigma(p)})$ for all $\sigma \in \mathfrak{S}_n$ such that $\chi_\pi(\sigma) \neq 0$. Consider permutations $(2345\dots) \dots (\dots)$ and $(2435\dots) \dots (\dots)$ (all cycles are the same except the first one, and for the first cycle, all numbers are the same except the first 4), from formula (6), we have:

$$(11) \quad na + \sum_{i,j>1} a_{ij} + a_{23} + a_{34} + a_{45} + E = 0$$

$$(12) \quad na + \sum_{i,j>1} a_{ij} + a_{24} + a_{43} + a_{35} + E = 0$$

for some linear combination E of a_{ij} 's. Thus

$$(13) \quad a_{23} + a_{34} + a_{45} = a_{24} + a_{43} + a_{35}$$

Similarly,

$$(14) \quad a_{ij} + a_{jk} + a_{km} = a_{ik} + a_{kj} + a_{jm}, \text{ for } i, j, k, m \text{ distinct.}$$

Next, consider permutations $(1234\dots k)\dots(\dots)$ and $(1254\dots k)\dots(\dots)$, again, from formula (7), we obtain:

$$(15) \quad na + a_{23} + a_{34} + E' - \sum_{i>1} a_{i,2} - \sum_{j>1} a_{k,j} = 0$$

$$(16) \quad na + a_{25} + a_{54} + E' - \sum_{i>1} a_{i,2} - \sum_{j>1} a_{k,j} = 0$$

Hence,

$$(17) \quad a_{25} + a_{54} = a_{23} + a_{34}$$

and thus

$$(18) \quad a_{ij} + a_{jk} = a_{ij'} + a_{j'k}, \text{ for all } i, j, j', k \text{ distinct.}$$

Now for $2 \leq i < j < k < m \leq n$, we have system of linear equations of the same form as Lemma 4.13. So we have relations:

$$\sum_{\substack{m \geq p \geq i \\ \mu(p) \neq p}} a_{p\mu(p)} = l\lambda_{ijkm}$$

where μ is a permutation of the set $\{i, j, k, m\}$, l is the number of elements moved by μ , and λ_{ijkm} is a constant number.

It is easy to see that λ_{ijkm} is the same for different choices of the set $\{2 \leq i < j < k < m \leq n\}$, for example, we can compare $\{i, j, k, m\}$ and $\{i, j, k, m'\}$ to obtain $\lambda_{ijkm} = \lambda_{ijk m'}$. From now on, we write all λ_{ijkm} 's as λ . Hence, given any permutation of the set $\{2, \dots, n\}$ moving l elements,

$$(19) \quad \sum_{\substack{n \geq p \geq 2 \\ \mu(p) \neq p}} a_{p\mu(p)} = l\lambda$$

Next, we find relations among the a_{ii} 's for $i \geq 2$. For this purpose, consider $\tau_1 = (243\dots k)\dots(\dots)$ and $\tau_2 = (253\dots k)\dots(\dots)$, then

$$(20) \quad na + \sum_{i,j>1} a_{ij} + a_{24} + \dots + a_{k2} + a_{55} + (\text{sum of } a_{ii} \text{'s for } i \neq 5 \text{ fixed by } \tau_1) = 0$$

$$(21) \quad na + \sum_{i,j>1} a_{ij} + a_{25} + \dots + a_{k2} + a_{44} + (\text{sum of } a_{ii} \text{'s for } i \neq 4 \text{ fixed by } \tau_1) = 0$$

Combine these two equations and equation (19) to obtain

$$a_{44} = a_{55}.$$

The same argument implies that $a_{ii} = a_{55}$ for all $n \geq i \geq 2$. Now we have:

$$\sum_{i,j>1} a_{ij} = \sum_{1 < i < j} (a_{ij} + a_{ji}) + 2(n-1)a_{55} = (n-1)(n-2)\lambda + 2(n-1)a_{55}$$

Let $\sigma = (1)$, formula(6) implies

$$(22) \quad na + (n-1)(n-2)\lambda + 3(n-1)a_{55} = 0$$

Let $\sigma = (2345\dots)\dots(\dots)(\sigma(1) = 1)$, again by formula (6)

$$(23) \quad na + ((n-2)(n-1) + l)\lambda + (3(n-1) - l)a_{55} = 0$$

where l is the number of elements moved by σ . Let $\sigma_1 = (123\dots p4)\dots(\dots)$ and $\sigma_2 = (143\dots p2)\dots(\dots)$. Then formula (7) gives:

$$(24) \quad 0 = na + (a_{23} + \dots + a_{p4}) + \tilde{E} - \sum_{i>1} a_{i2} - \sum_{j>1} a_{4j}$$

$$(25) \quad 0 = na + (a_{43} + \dots + a_{p2}) + \tilde{E} - \sum_{i>1} a_{i4} - \sum_{j>1} a_{2j}$$

Note that \tilde{E} comes from the product of disjoint cycles in σ_1 and σ_2 except the first one, so they are indeed the same, and if we assume that σ_1 moves l' elements, and the first cycle in σ_1 moves r elements, then $\tilde{E} = (l' - r)\lambda + (n - l')a_{55}$. On the other hand,

$$a_{23} + \dots + a_{p4} = a_{23} + \dots + a_{p4} + a_{42} - a_{42} = r\lambda - a_{42}$$

$$a_{43} + \dots + a_{p2} = a_{43} + \dots + a_{p2} + a_{24} - a_{24} = r\lambda - a_{24}$$

Equations (24) and (25) gives:

$$(26) \quad na + (l' + 2n - 5)\lambda + (n - l' + 2)a_{55} = 0$$

Now equations (22), (23), and (26) imply that $\lambda = a = a_{55} = 0$. From equation (24), we have:

$$a_{42} = \sum_{j>1} a_{4j} + \sum_{i>1} a_{i2}$$

similarly,

$$a_{k2} = \sum_{j>1} a_{kj} + \sum_{i>1} a_{i2} \text{ for all } n \geq k \geq 2$$

The sum of these equations is:

$$\sum_{i>1} a_{i2} = (n-1) \sum_{i>1} a_{i2} + \sum_{i,j>1} a_{ij} = (n-1) \sum_{i>1} a_{i2}.$$

Hence $\sum_{i>1} a_{i2} = 0$. For the same reason $\sum_{j>1} a_{4j} = 0$, therefore $a_{42} = 0$. By the same argument, $a_{ij} = 0$ for all $n \geq i \neq j \geq 2$, and this completes the proof of the lemma. \square

Lemma 4.15. *If $n \geq 6$, then for any partition λ of n , except $\lambda = (3, 1, 1, 1)$, $\lambda = (4, 1, 1)$ and $\lambda = (4, 1, 1, 1)$, there exists a permutation $\tau \in \mathfrak{S}_n$ satisfying conditions (1), (2), (3) in Lemma 4.14.*

Proof. Write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ where $\lambda_1 \geq \lambda_2 \geq \dots \lambda_p \geq 1$ and $\sum_{i=1}^p \lambda_i = n$. Without loose of generality, we may assume $p \geq \lambda_1$, otherwise, we can consider the conjugate λ' of λ . There exists a largest integer m such that the Young diagram of λ contains an $m \times m$ square.

Now we will construct τ using the Murnagham-Nakayama Rule (See [?])case by case:

- (1) If $m = 1$ then λ is a hook: $(\lambda_1, 1, \dots, 1)$, there are the following cases:
 - (a) $p > \lambda_1$ and $\lambda_1 \geq 4$. Take $\tau = (p-1, 1^{n-p+1})$ then $\chi_\lambda(\tau) \neq 0$ by the Murnagham-Nakayama Rule. In this case, $n \geq 8$.
 - (b) $p > \lambda_1$ and $\lambda_1 = 1$. This case is trivial.
 - (c) $p > \lambda_1$ and $\lambda_1 = 2$ or 3 . $\tau = (4, 1^{n-4})$ will work.
 - (d) $p = \lambda_1$. Take $\tau = (p-1, 1^{n-p+1})$ if $p \geq 6$ is even and $\tau = (p-2, 1^{n-p+2})$ if $p \geq 7$ is odd. In this case $n \geq 11$.
 - (e) $p = \lambda_1 = 5$. τ exists by checking the character tables.
- (2) If $m \geq 2$, let ξ be the length of the longest skew hook contained in the young diagram of λ . Then take $\tau = (\xi, 1^{n-\xi})$.

\square

proof of Theorem 1.5. For the case $n = 5$, one can check directly. By Lemma 4.14 and lemma 4.15, we know that for $n \geq 6$ and π not equal to $(3, 1, 1, 1)$ and $(4, 1, 1, 1)$, the lie algebra of $\{C \in M_{n \times n} \mid P_\pi(C * X) = P_\pi(X)\}$ is $T(E) \otimes Id_F \oplus Id_E \otimes T(F)$, so the identity component of $\{C \in M_{n \times n} \mid P_\pi(C * X) = P_\pi(X)\}$ is $T(GL(E) \times GL(F))$, and hence the identity component of $G(P_\pi)$ is $\Delta(\mathfrak{S}_n) \ltimes T(GL(E) \times GL(F)) \ltimes \mathbb{Z}_2$. For cases $\pi = (3, 1, 1, 1)$, $\pi = (4, 1, 1)$ the statement is true by Theorem 1.7. \square

By investigating the equation (3), we can give a sufficient condition for the stabilizer of P_π to be $\Delta(\mathfrak{S}_n) \ltimes T(GL(E) \times GL(F)) \ltimes \mathbb{Z}_2$ as follows:

Lemma 4.16. *Let π be a partition of n which is not $(1, \dots, 1)$ or (n) . Assume that there exist permutations $\sigma, \tau \in \mathfrak{S}_n$ and an integer $p \geq 2$, such that $\chi_\pi((i_1 \dots i_p)\sigma) \neq 0$, $\chi_\pi((1i_1 \dots i_p)\sigma) \neq 0$, $\chi_\pi(\tau) \neq 0$ and $\chi_\pi((ij)\tau) \neq 0$, where $(i_1 \dots i_p)$ and $(1i_1 \dots i_p)$ are cycles disjoint from σ , and (ij) is disjoint from τ . Then the stabilizer of P_π is $\Delta(\mathfrak{S}_n) \ltimes T(GL(E) \times GL(F)) \ltimes \mathbb{Z}_2$.*

Proof. For convenience, we will show for the case $\sigma = (1)$ and $p = 2$, the other cases are similar. In equations 3, let $\sigma = (ij)$ and $(1ij)$, where $1 < i, j \leq n$ and $i \neq j$. Then

$$(27) \quad c_{ji}c_{ij}c_{11} \prod_{k \neq 1, i, j} c_{kk} = 1$$

$$(28) \quad c_{ji}c_{1j}c_{i1} \prod_{k \neq 1, i, j} c_{kk} = 1$$

So $c_{ij} = \frac{c_{i1}c_{1j}}{c_{11}}$. Similarly, the existence of τ will give the relation $c_{ii} = \frac{c_{i1}c_{1i}}{c_{11}}$ for all $1 \leq i \leq n$. Set

$$\mathbf{A} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{pmatrix}$$

Where $a_{ii} = c_{i1}$ and $b_{jj} = \frac{c_{1j}}{c_{11}}$. So we have

$$C * X = AXB, \text{ with } \det(AB) = 1.$$

□

The following two propositions guarantee the existence of permutations satisfied conditions in Lemma (4.16).

Proposition 4.17. *Let $n \geq 3$, and π be a non-symmetric partition of n , then there exists nonnegative integers k_1, \dots, k_r such that $k_1 + \dots + k_r = n - 2$, such that $|\chi_\pi(\tau)| = 1$, where τ is a permutation of type $(k_1, \dots, k_r, 1^2)$ or $(k_1, \dots, k_r, 2)$.*

Proof. See Proposition (3.1), Coelho, M. Purificação and Duffner, M. Antónia [?]. □

Proposition 4.18. *Let $n > 4$ and π be a non-symmetric partition of n , then there exists nonnegative integers k_1, \dots, k_r and q with $k_r > 1$, $q \geq 1$, and $k_1 + \dots + k_r + q = n$, such that $\chi_\pi(\sigma) \neq 0$, where $\sigma \in \mathfrak{S}$ is of type $(k_1, \dots, k_r, 1^q)$ or $(k_1, \dots, k_r + 1, 1^{q-1})$.*

Proof. See Proposition (3.2), Coelho, M. Purificação and Duffner, M. Antónia [?]. □

proof of Theorem 1.7. Since $n \geq 5$, by propositions 4.17 and 4.18, there exist permutations satisfying conditions in Lemma 4.16, then the theorem follows. □

Acknowledgement

I would like to thank the anonymous referee for his/her valuable comments.

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